Multi-Soliton Solution of NLSE using Darboux Transformation.

Raja Mohsin Hameed

Dr. Waqas Masood (CUI, Islamabad Pakistan) & Dr. Mohsin Siddiq (NCP Islamabad Pakistan)

Introduction

We shall obtain multi-soliton solutions of Nonlinear Schrodinger Equation (NLE) by using Darboux transformation (DT).

- In 1882, G. Darboux proposed a transformation method to solve the **linear differential equations**, known as **DT**.
- Latter on V. B. Matveev and M. A. Salle introduced a method to solve integrable **NLEs** with the help of **Lax pair (linear system)**, to determine the **multi-soliton solutions** by using known solutions.
- Widely used to get multi-soliton solutions of KdV, mKdV, NLS & Sine Gordon equations. Etc.
- Darboux Transformation can also be used to find breather and Rogue wave solutions after few modifications.

Method: Darboux Transformation

We derive Darboux Transformation for one dimensional NLSE with constant coefficients P and Q, as given below.

$$\iota\Psi_t + P\Psi_{xx} - 2Q\Psi \mid \Psi \mid^2 = 0$$

Its corresponding linear system (lax pair) of AKNS scheme is

$$\Psi_x = U\Psi, \quad \text{and} \quad \Psi_t = V\Psi, \quad \text{where}$$

$$U = \begin{pmatrix} -\frac{i}{2}\lambda & \sqrt{\frac{Q}{P}}r \\ \sqrt{\frac{Q}{P}}q & \frac{i}{2}\lambda \end{pmatrix} = U_o + \lambda U_1$$

$$V = \begin{pmatrix} \frac{iP}{2}\lambda^2 + iQqr & -\sqrt{QP}r\lambda - i\sqrt{QP}r_x \\ -\sqrt{QP}q\lambda + i\sqrt{QP}q_x & -\frac{iP}{2}\lambda^2 - iQqr \end{pmatrix}$$

$$V = V_0 + \lambda V_1 + \lambda^2 V_2$$

- Zero curvature condition $U_t - V_x + [U,V] = 0$ verifies that linear system is compatible to the Nonlinear system.
- We get system of NLSE (2 equations), which reduces to NLSE when q=r^*.

$$\iota q_t + Pq_{xx} - 2Qq \mid q \mid^2 = 0$$

N-Fold Darboux Transformation

Darboux Transformation is a special gauge transformation, as

$$\Psi[N] = T(x,t,\lambda)\Psi$$

Then transformed linear system is

$$\Psi_x[N] = U[N]\Psi[N] \quad \text{Where} \quad U[N] = (T_x + TU)T^{-1}$$

and

$$\Psi_t[N] = V[N]\Psi[N] \quad \text{Where} \quad V[N] = (T_t + TV)T^{-1}$$

$$T(x,t,\lambda) = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & D(x,t,\lambda) \end{pmatrix} = \sum_{J=0}^{N} a_j(x,t)\lambda^N$$

$$a_j = \begin{pmatrix} A_j(x,t) & B_j(x,t) \\ C_j(x,t) & D_j(x,t) \end{pmatrix} (j = 0,1,2,\ldots,N-1) \quad \text{and} \quad a_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since det *T*(*x, t, λ*) has 2*N* zeros λ_k and the coefficient of λ^N is an identity matrix and from transformation equation we get

$$\sum_{j=0}^{N-1}[A_j + \beta_k B_j]\lambda_k^j = -\lambda_k^N$$

$$\sum_{j=0}^{N-1}[D_j + \alpha_k C_j]\lambda_k^j = -\lambda_k^N$$

Where

$$\beta_k = \frac{1}{\alpha_k} = \frac{\phi_2(\lambda_k) - b_k\psi_2(\lambda_k)}{\phi_1(\lambda_k) - b_k\psi_1(\lambda_k)}$$

From these two equation, we can get values of A, B, C and D of matrix T by using Cramer's rule.

From equation $U[N] = (T_x + TU)T^{-1}$ and $V[N] = (T_t + TV)T^{-1}$
We get new protentional in form of q[N] and r[N].

$$r[N] = r + i\sqrt{\frac{Q}{P}}B_{N-1} \quad \text{and} \quad q[N] = q + i\sqrt{\frac{Q}{P}}C_{N-1}$$

We can obtain the values of $B_{N-1}$and $C_{N-1}$by using Cramer's rule for any value of N. Hence, we obtain N Soliton Solutions of NLSE in the form of q[N] and r[N].

By introducing the reduction condition $q[N] = r^*[N]$ and following assumptions proposed by Neugebauer.

$$\lambda_{2j} = \lambda_{2j-1}^*, \quad \alpha_{2j} = -\frac{1}{\alpha_{2j-1}^*}, \quad \beta_{2j} = -\frac{1}{\beta_{2j-1}^*}$$

Result: Multi-Soliton Solutions

The matrix given below is multi-soliton solution for NLSE when seed solution q=0.
For any value of N, it gives exact soliton solution

$$q[N] = q + \iota\sqrt{\frac{P}{Q}} \frac{\begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^N \\ 1 & -(\alpha_1^*)^{-1} & \lambda_1^* & -\lambda_1^*(\alpha_1^*)^{-1} & \cdots & \lambda_1^{*N-1} & \lambda_1^{*N} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & -(\alpha_N^*)^{-1} & \lambda_N^* & -\lambda_N^*(\alpha_N^*)^{-1} & \cdots & \lambda_N^{*N-1} & \lambda_N^{*N} \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \lambda_1\alpha_1 & \cdots & \lambda_1^{N-1} & \lambda_1^{N-1}\alpha_1 \\ 1 & -(\alpha_1^*)^{-1} & \lambda_1^* & -\lambda_1^*(\alpha_1^*)^{-1} & \cdots & \lambda_1^{*N-1} & -\lambda_1^{*N-1}(\alpha_1^*)^{-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & -(\alpha_N^*)^{-1} & \lambda_N^* & -\lambda_N(\alpha_N^*)^{-1} & \cdots & \lambda_N^{*N-1} & -\lambda_N^{N-1}(\alpha_N^*)^{-1} \end{vmatrix}}$$

Result: Single Soliton Solution

- For N=1
- We choose q=0 as seed solution, then equation of muli-soliton solutions will become.

$$q[1] = q + \iota\sqrt{\frac{P}{Q}} \frac{\begin{vmatrix} 1 & \lambda_1 \\ 1 & \lambda_1^* \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 \\ 1 & -(\alpha_1^*)^{-1} \end{vmatrix}}$$

$$\alpha_k = \frac{-1}{b_k}\exp(-\iota\lambda_k x + \iota P\lambda_k^2 t)$$

$$b_1 = \exp[\mu_1 + \iota\sigma_1]$$

$$\lambda_1 = \xi_1 + \iota\eta_1$$

$$q[1] = \sqrt{\frac{P}{Q}}\eta_1 \sec h(X_1)\exp(\iota Y_1)$$

$$Y_1 = \xi_1 x - P(\xi_1^2 - \eta_1^2)t + \sigma_1$$

$$X_1 = \eta_1 x - 2P\xi_1\eta_1 t - \mu_1$$

Conclusion

- To find the N-Fold Darboux transformation of NLSE, we have applied a method proposed by G. Neugebauer using Lax pairs in matrix form.
- The Lax pairs provided the compatible linearized system for the nonlinear Schrödinger equation.
- We have derived N-fold DT of NLSE system by using gauge transformation and reduction technique.
- Later, we have obtained the multi-soliton solutions by choosing

i. Zero solution as initial condition.

ii. Complex eigenvalue.

- The single soliton solution contains product of sec-hyperbolic and exponential function that appears as an envelope soliton structure

Future Work

- We shall derive N-fold Darboux transformations of nonlinear soliton equations using different approaches of Darboux transformation and find the multi-soliton solutions of different plasma models.
- To observe the spontaneous modulation of the wave, we shall find the multi-rogue wave solutions using Darboux transformation and modified Darboux transformation.
- The graphical descriptions will be shown for different plasma models.
- We shall find the periodic multi-rogue wave solutions of different NLEEs.

Acknowledgement

First and foremost, I wish to express my sincere gratitude to my supervisor **Prof. Waqas Masood (CUI Pakistan)** and Co-supervisor **Dr. Mohsin Siddiq (NCP Islamabad Pakistan)**, for both providing necessary background and investigating insight to explore the fascinating concepts of Plasma physics, Darboux Transformation.